CSS322: Scientific Computing

Final Mock Exam

curated by The Peanuts

Name.............ID.............Section.............Seat No.............

Conditions: Semi-closed Book

Directions:

1. This exam contains 12 pages (including this one). If yours has fewer, congratulations, you've discovered truncation error.
2. Write your name clearly at the top.
3. Show your work. Partial credit is real, this isn't binary classification.
4. Answers must be written in English. Python, MATLAB, or Morse code will not be graded (though we'll admire the commitment).
5. You may use your cheat sheets. You may not use your neighbor's memory, GPU, or ChatGPT API key.
6. If your solution diverges, you may try reducing your step size or breathing rate.

*For solution, **click here**.*

Question 1

Suppose that the centered difference approximation to the derivative of a function at a given point produces the value 8.5 for $h = 0.4$ and the value 9.2 for $h = 0.2$. Use Richardson extrapolation to obtain a better approximate value for the derivative.

Question 2

Given three data points $(x_i, y_i) = (1, 3), (2, 5), (3, 9)$. We want to fit these data points with the function $y = c_1 x + c_2 x^2$.

1. Find such a function that best fits the given data points in least-squares sense using the method of normal equations.

2. Do the same but with Householder transformation instead. Show your work for finding the first Householder vector v_1 and computing $H_1 A$.

Question 3

Consider solving $\cos(x) = x$ using Newton's method. Start with $x^{(0)} = 1$, perform two steps of Newton's method to find $x^{(2)}$.

Question 4

Consider solving $e^x - 2x - 1 = 0$ using the secant method. Start with $x^{(0)} = 0$ and $x^{(1)} = 0.5$, perform one step of the secant method to find $x^{(2)}$.

Question 5

Carry out one iteration of Newton's method applied to the system of nonlinear equations

$$x_1^2 - x_2^2 = 0,$$
$$2x_1x_2 = 1,$$

with starting value $x^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Question 6

Consider minimizing the function $f(x) = x^3 - 2x + 1$. Perform one step of Newton's method for optimization to find $x^{(1)}$ using $x^{(0)} = 1.5$ as the initial guess.

Question 7

Consider the following initial value problem (IVP):

$$\frac{dy}{dt} = 2t - y; \quad y(0) = 1.$$

1. Perform one step of the Backward Euler method using step size $h = 0.5$ to compute t_1 and y_1.

2. Using (t_1, y_1) found from part (a), and additionally $y_0 = 1$, perform one step of Adams-Bashforth second order (AB2) to compute (t_2, y_2) using the same step size.

Question 8

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & -2 & 3 \end{bmatrix}.$$

Suppose the SVD of A is $A = U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} 5.2915 & 0 & 0 & 0 \\ 0 & 3.8637 & 0 & 0 \\ 0 & 0 & 2.1749 & 0 \end{bmatrix}.$$

1. What is $\|A\|_2$?

2. What is the rank of A?

3. If we wanted to find the best rank-2 approximation of A using SVD, what would be the approximation error in the 2-norm?

Question 9

Suppose a matrix $A \in \mathbb{R}^{4\times 4}$ has the following eigenvalues: $-25, -5, 8, 20$.

1. Which eigenvalue, if any, corresponds to the eigenvector that power method would converge to? Justify your answer.

2. Which eigenvalue, if any, corresponds to the eigenvector that inverse power method would converge to? Justify your answer.

3. Which eigenvalue, if any, corresponds to the eigenvector that inverse shifted power method with shift $\sigma = 10$ would converge to? Justify your answer.

4. Suppose we apply one iteration of power method starting with $v^{(0)} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$. If after normalization we get $\mu^{(1)} = 15.5$, what does this tell us about $v^{(0)}$?

Question 10

Solve the BVP

$$u'' = -2 + u, \quad 0 < t < 1,$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 1$$

using the collocation method with one interior collocation point $t_2 = 0.5$ for an approximate solution of the form

$$u(t) \approx v(t) = x_1 + x_2 t + x_3 t^2.$$

Question 11

Consider the heat equation on the domain $[0, 1]$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = 0$, $u(1,t) = 0$ and initial condition $u(x, 0) = \sin(\pi x)$.

1. Using the finite difference method with spatial step $h = 0.25$ and temporal step $k = 0.01$, write down the explicit finite difference scheme for approximating $u(0.25, 0.01)$. You do not need to compute the numerical value.

2. What is the stability condition for the explicit method in terms of h and k? Is the choice of $h = 0.25$ and $k = 0.01$ stable?